Numerical Calculation of Fields

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INSTITUT FÜR HOCHFREQUENZTECHNIK UND ELEKTRONIK

Antenna Modeling

- To model antennas' behavior, the structure has to be meshed and the Maxwell's equations solved

$$\mathrm{rot}\ \vec{H} = \vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}$$

$$\mathrm{rot}\ \vec{E} = \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} - \vec{M}$$

$$\mathrm{div}\ \vec{B} = \vec{\nabla} \cdot \vec{B} = \rho_m$$

$$\mathrm{div}\ \vec{D} = \vec{\nabla} \cdot \vec{D} = \rho_e$$

?

Maxwell‘s Equations

Differential form: **Integral form:**

$$\mathrm{rot}\ \vec{H} = \vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J} \quad \Leftrightarrow \quad \oint_{\partial A} \vec{H} \cdot d\vec{s} = \iint_{A} \frac{d\vec{D}}{dt} \cdot d\vec{A} + \iint_{A} \vec{J} \cdot d\vec{A}$$

$$\mathrm{rot}\ \vec{E} = \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} - \vec{M} \quad \Leftrightarrow \quad \oint_{\partial A} \vec{E} \cdot d\vec{s} = -\iint_{A} \frac{d\vec{B}}{dt} \cdot d\vec{A} - \iint_{A} \vec{M} \cdot d\vec{A}$$

$$\mathrm{div}\ \vec{B} = \vec{\nabla} \cdot \vec{B} = \rho_m \quad \Leftrightarrow \quad \oiint_{\partial V} \vec{B} \cdot d\vec{F} = \iiint_{V} \rho_m \cdot dV$$

$$\mathrm{div}\ \vec{D} = \vec{\nabla} \cdot \vec{D} = \rho_e \quad \Leftrightarrow \quad \oiint_{\partial V} \vec{D} \cdot d\vec{F} = \iiint_{V} \rho_e \cdot dV$$

Nabla operator:

$$\vec{\nabla} = \frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y + \frac{\partial}{\partial z}\vec{e}_z$$

$$\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{A}\right) = 0$$

Continuity equation:

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho_e}{\partial t}$$

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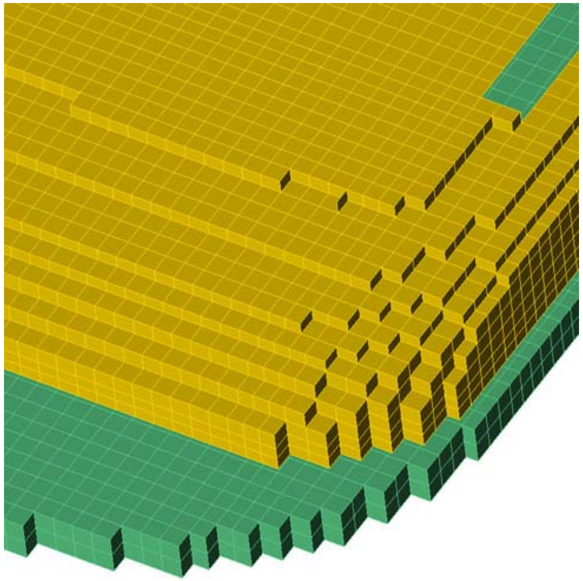
Time Domain Methods

- FDTD/FIT
  - FDTD – Finite Difference Time Domain
  - FIT – Finite Integration Technique (variation of FDTD used in CST)
  - Ideal for highly inhomogeneous materials
  - Pulse excitation makes this method perfect for wideband problems
  - The basic algorithm proposed by Yee in 1966 [1]
  - Structure and surrounding space discretized into cubic mesh



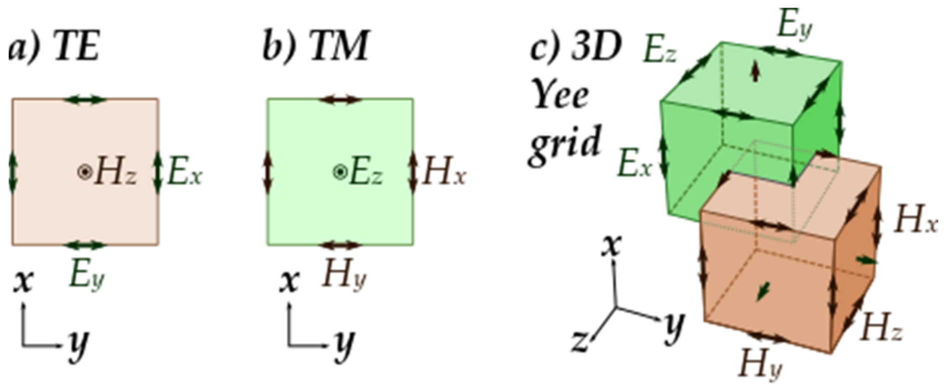
FDTD approach has an excellent scaling performance as the problem size grows. As the number of unknowns increases, the FDTD approach quickly outpaces other methods in efficiency.

Finite Integration Technique - Algorithm

1. Discretize space and time so that the electric and magnetic fields are staggered in both space and time.
2. Solve the resulting integral equations to obtain “update equations” that express the (un-known) future fields in terms of (known) past fields.
3. Evaluate the magnetic fields one time-step into the future so they are now known (effectively they become past fields).
4. Repeat the previous two steps until the fields have been obtained over desired duration.

Volume Discretization

- The structure is discretized in cubic cells (Yee grid)
- In one lattice the electric field vector components are represented on the edges of the cube and magnetic field components form the normal to the faces of the cubes
- In the other lattice the magnetic field vector components are represented on the edges of the cube and electric field components form the normal to the faces of the cubes



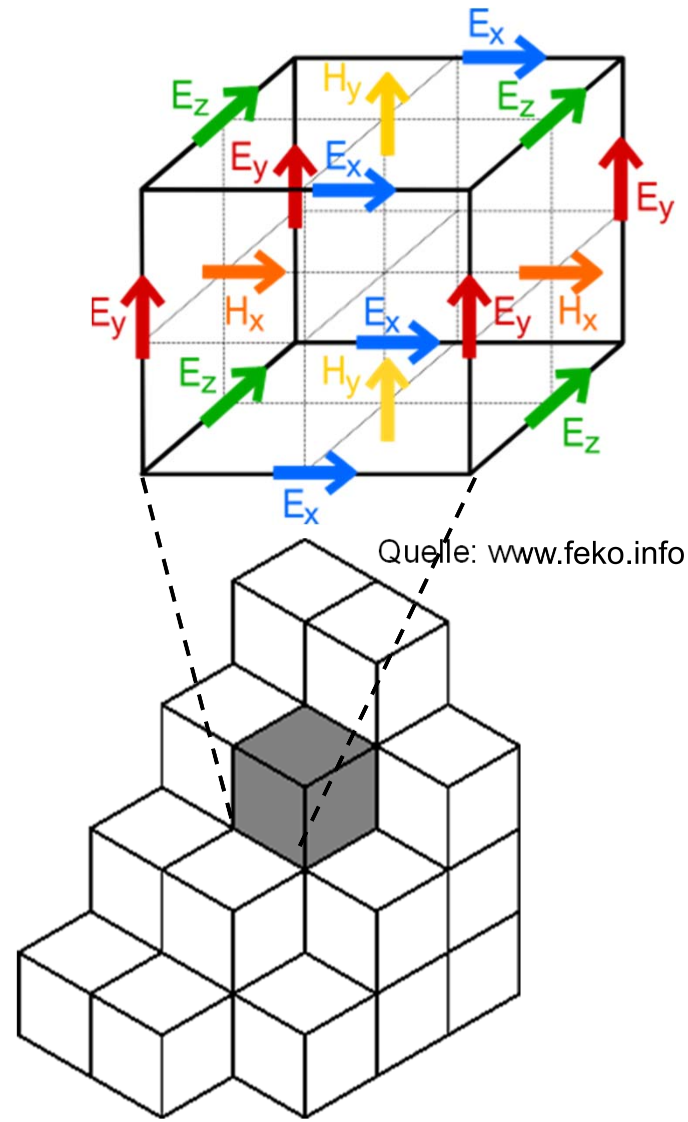
Finite Integration Technique - Calculation

- Calculation for TE cells

$$\oint_L \vec{E}\cdot d\vec{s} = -\iint_A \frac{\partial \vec{B}}{\partial t}\cdot d\vec{A}$$

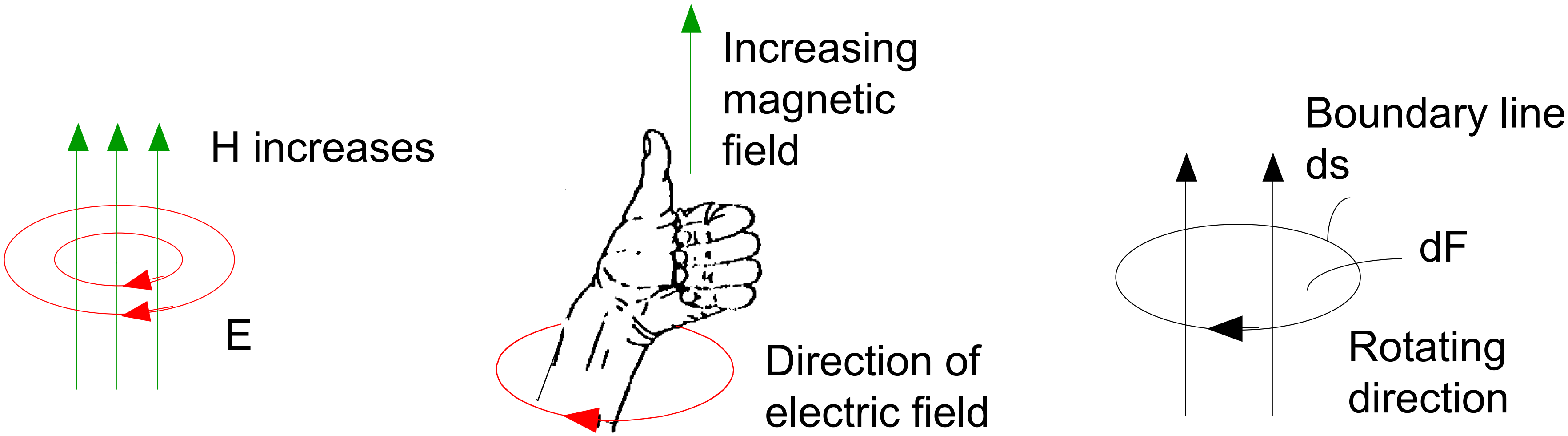
$$\hat{\hat{b}} = \iint_A \vec{B}\cdot d\vec{A}$$

$$\hat{e} = \oint_L \vec{E}\cdot d\vec{s}$$

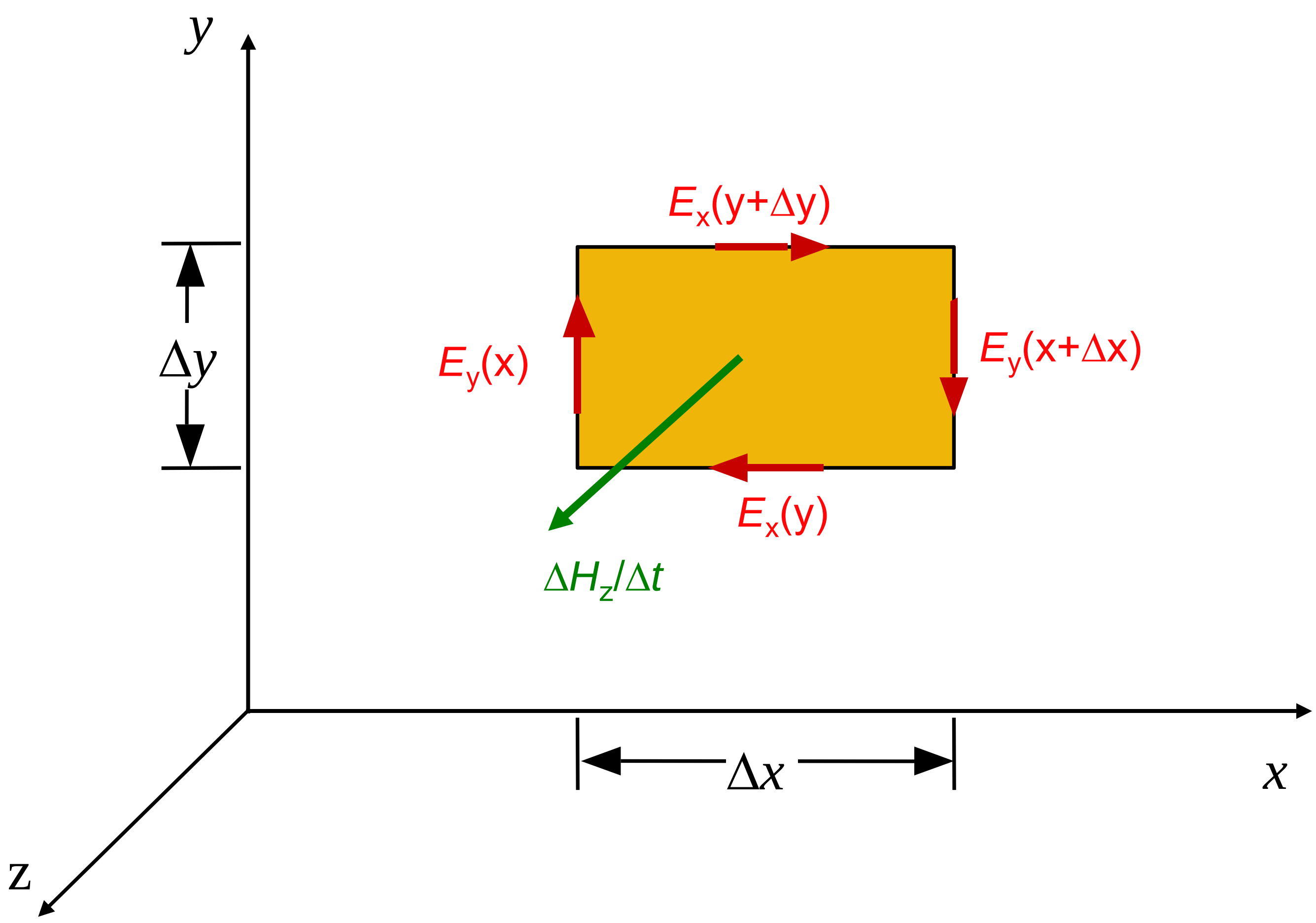


Quelle: www.feko.info

Faraday's Law of Induction



Derivation of the Field Components (Faraday's Law of Induction)



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Calculation of Faraday's Law

$$\oint_{\partial A} \vec{E} \cdot d\vec{s} = -\iint_A \frac{d\vec{B}}{dt} \cdot d\vec{F}$$

$$E_y(x)\Delta y + E_x(y+\Delta y)\Delta x - E_y(x+\Delta x)\Delta y - E_x(y)\Delta x = -\frac{\Delta B_z}{\Delta t}\Delta x \cdot \Delta y$$

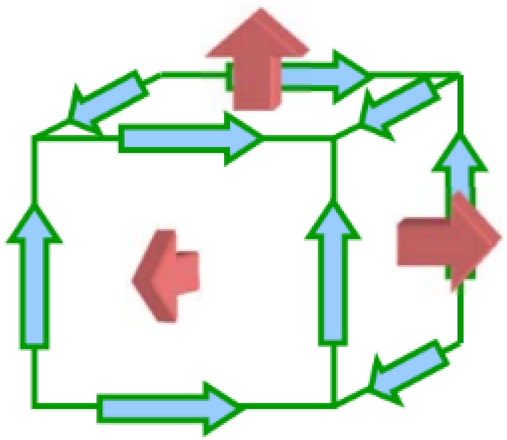
$$\frac{E_y(x) - E_y(x+\Delta x)}{\Delta x} - \frac{E_x(y) - E_x(y+\Delta y)}{\Delta y} = -\frac{\Delta B_z}{\Delta t}$$

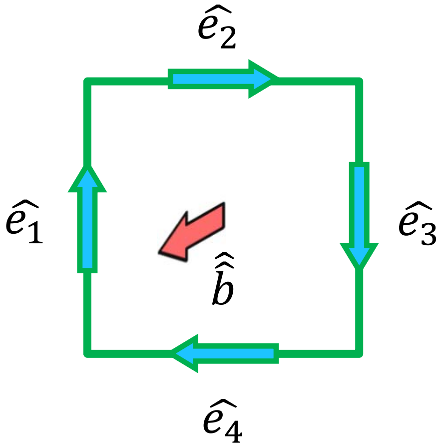
$\Delta x \to 0 \qquad \Delta y \to 0$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\partial B_z}{\partial t} \qquad \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t} \qquad \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t}$$

For three dimensions, a system of first-order differential equations for E-field (and H-field) is established.

Numerical Calculation – TE cells





$$\oint_L \vec{E} \cdot d\vec{s} = -\iint_A \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A}$$

$$\hat{\hat{b}} = \iint_A \vec{B} \cdot d\vec{A}$$

$$\hat{e} = \oint_L \vec{E} \cdot d\vec{s}$$

$$\widehat{e_1} + \widehat{e_2} - \widehat{e_3} - \widehat{e_4} = -\frac{d}{dt}\hat{\hat{b}}$$

$$\sum_{i=1,4} c_i \widehat{e_i} = -\frac{d}{dt}\hat{\hat{b}}$$

$$c_i = +/-1$$

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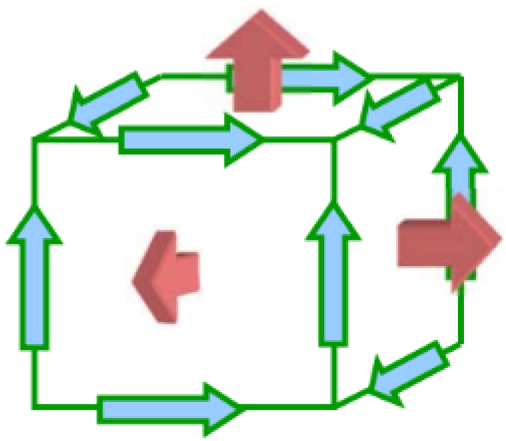
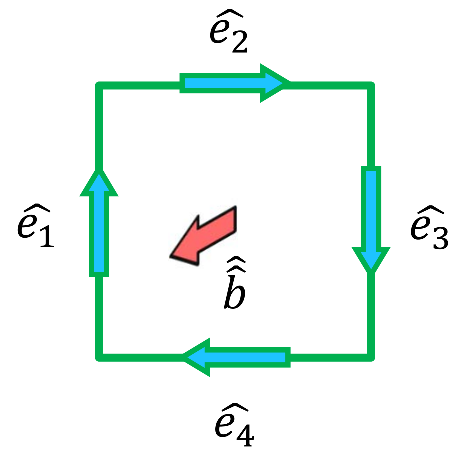
Numerical Calculation – TE cells

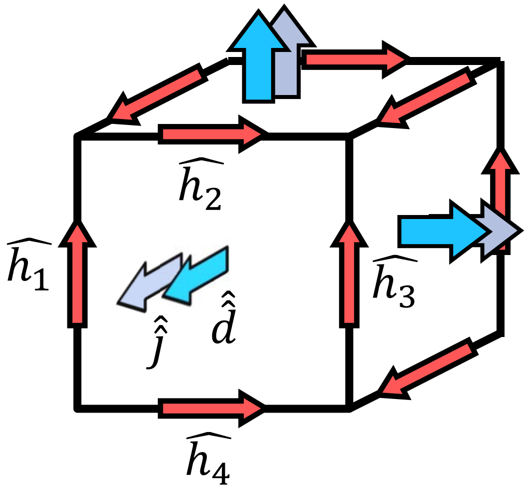




$$\sum_{i=1,4} c_i \widehat{e_i} = -\frac{d}{dt}\widehat{\widehat{b}} \quad \Rightarrow \quad C\hat{e} = -\frac{d}{dt}\widehat{\widehat{b}}$$

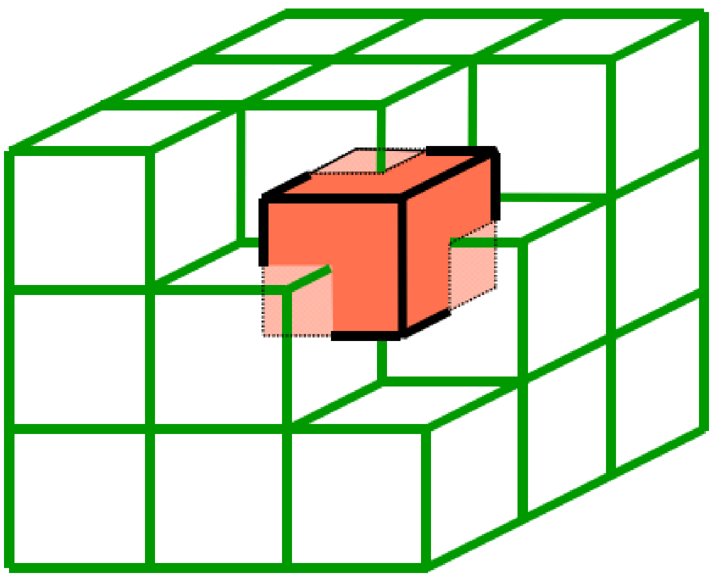
$$C = \begin{pmatrix} 1+1-1-1 & \cdots & \\ \vdots & \ddots & \vdots \\ & \cdots & \end{pmatrix} \qquad \hat{e} = \begin{pmatrix} \widehat{e_1} \\ \vdots \\ \widehat{e_N} \end{pmatrix} \qquad \widehat{\widehat{b}} = \begin{pmatrix} \widehat{\widehat{b_1}} \\ \vdots \\ \widehat{\widehat{b_N}} \end{pmatrix}$$

Numerical Calculation – TM cells



$$\oint_L \vec{H} \cdot d\vec{s} = \iint_A \frac{\partial \vec{D}}{\partial t} \cdot d\vec{A} + \iint_A \vec{J} \cdot d\vec{A}$$

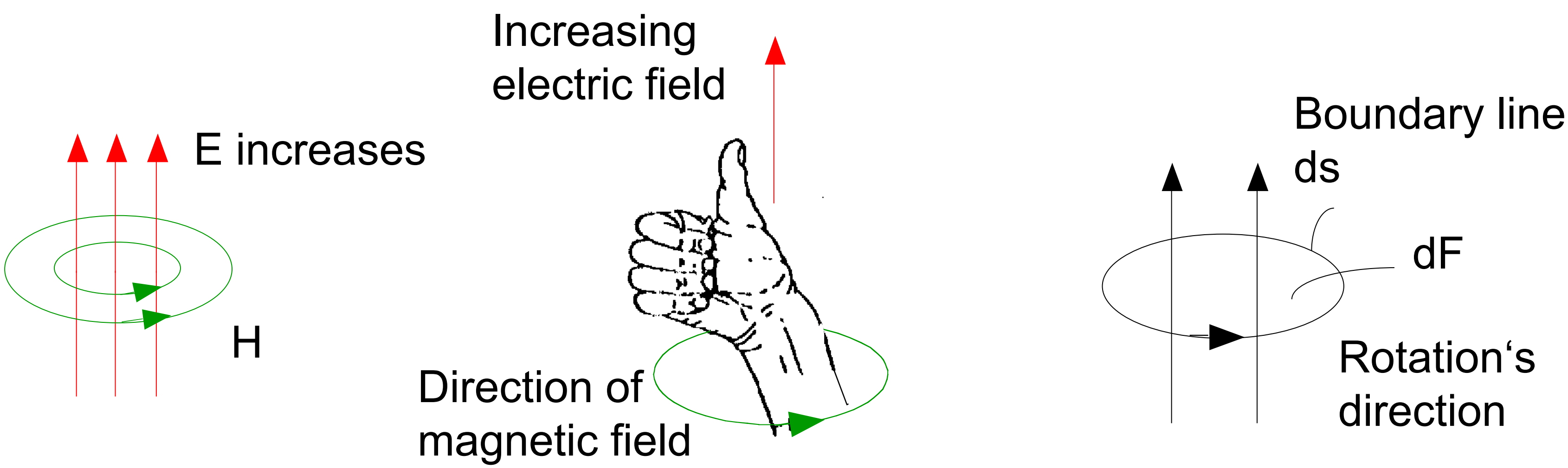
$$\hat{\hat{d}} = \iint_A \vec{D} \cdot d\vec{A}$$



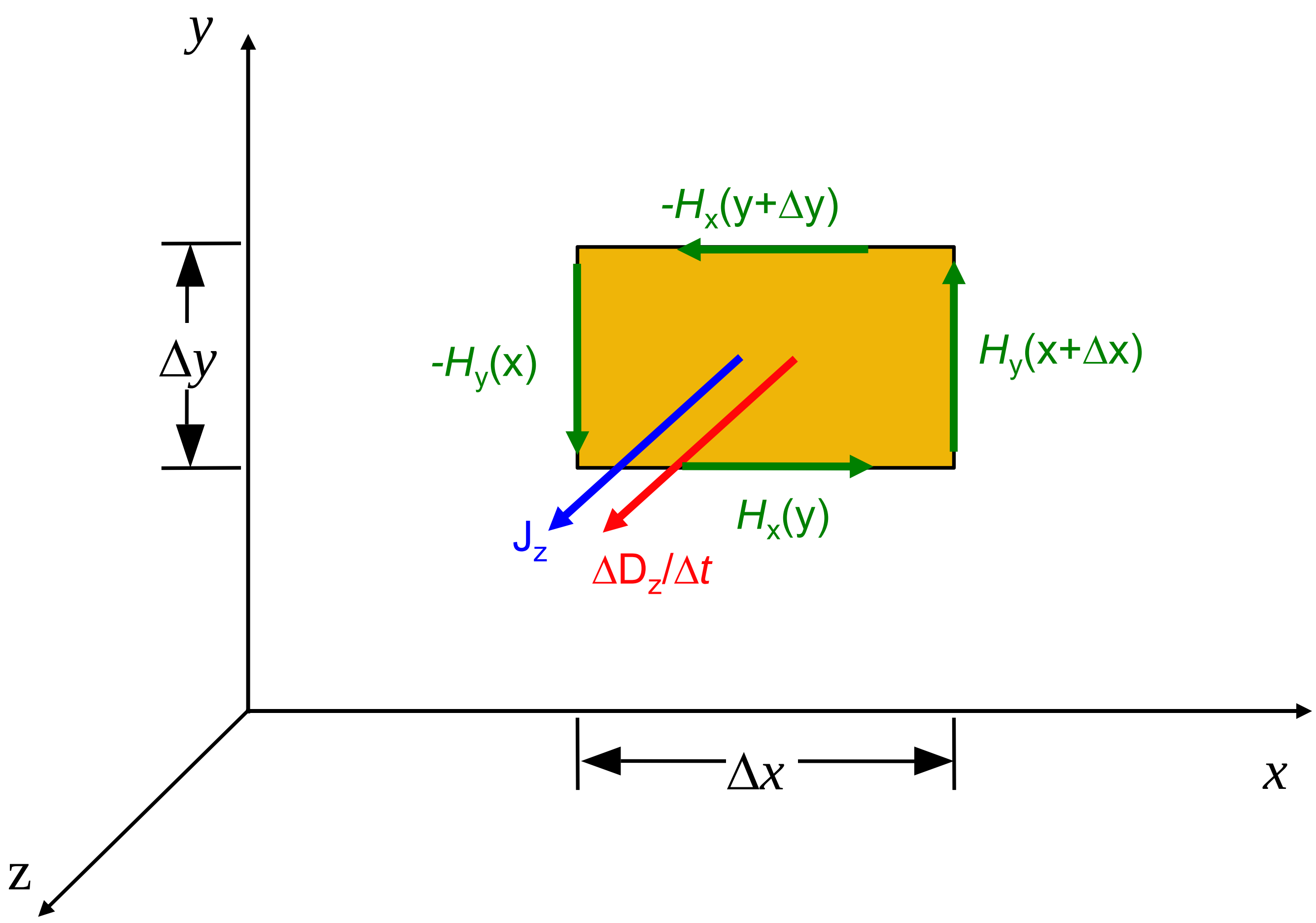
$$\hat{\hat{j}} = \iint_A \vec{J} \cdot d\vec{A}$$

$$\hat{h} = \oint_L \vec{H} \cdot d\vec{s}$$

Ampère's Circuital Law



Derivation of the Field Components (Ampère's Circuital Law)



Calculation of Ampère's Law

$$\oint_{\partial A} \vec{H} \cdot d\vec{s} = \iint_A \frac{d\vec{D}}{dt} \cdot d\vec{A} + \iint_A \vec{J} \cdot d\vec{A}$$

$\vec{D}$ Electric displacement field
$\vec{J}$ Electric current density

$$-H_y(x)\Delta y - H_x(y+\Delta y)\Delta x + H_y(x+\Delta x)\Delta y + H_x(y)\Delta x = \frac{\Delta D_z}{\Delta t}\Delta x \cdot \Delta y + J_z \Delta x \cdot \Delta y$$

$$\frac{H_y(x+\Delta x) - H_y(x)}{\Delta x} - \frac{H_x(y+\Delta y) - H_x(y)}{\Delta y} = \frac{\Delta D_z}{\Delta t} + J_z$$

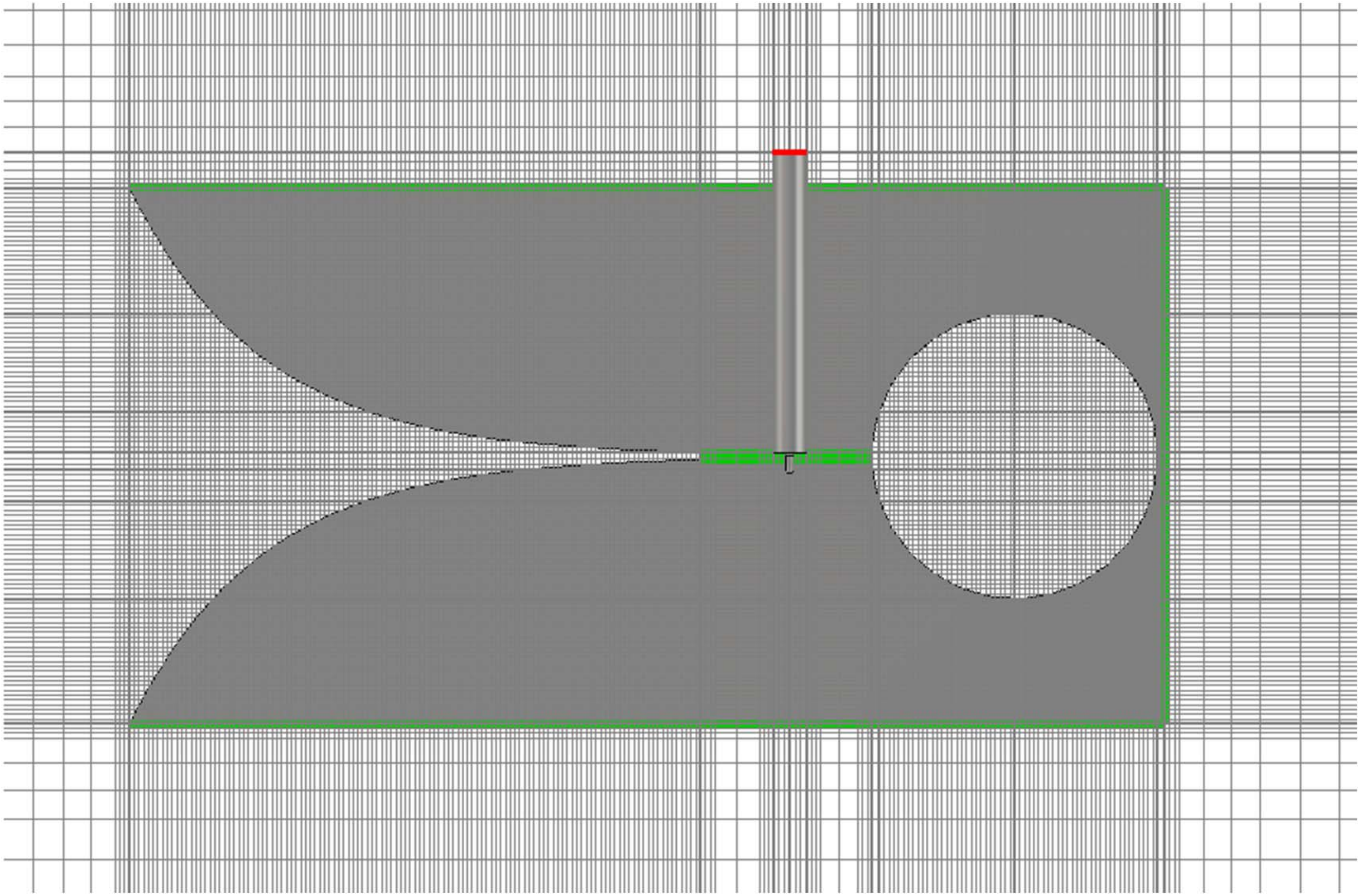
$\Delta x \to 0 \quad \Delta y \to 0$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \frac{\partial D_z}{\partial t} + J_z \qquad \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = \frac{\partial D_x}{\partial t} + J_x \qquad \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = \frac{\partial D_y}{\partial t} + J_y$$

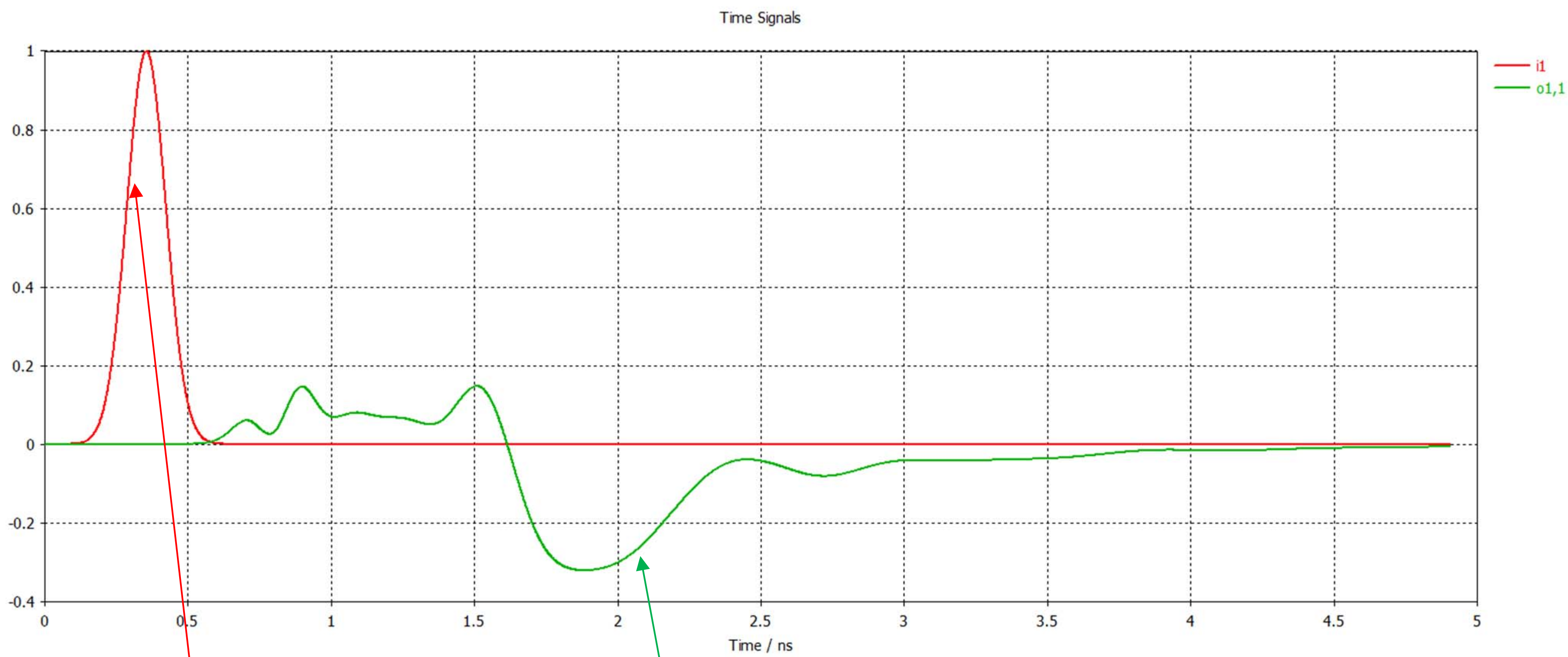
Continuity equation:

$$\vec{\nabla} \cdot \vec{\nabla} \times \vec{H} = 0 = \vec{\nabla} \cdot \frac{\partial \vec{D}}{\partial t} + \vec{\nabla} \cdot \vec{J} = \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} \quad \Rightarrow \quad \vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

Time Domain Method – Example Meshing



Time Domain Method – Example Impulse Answer



Excitation impulse

System’s answer

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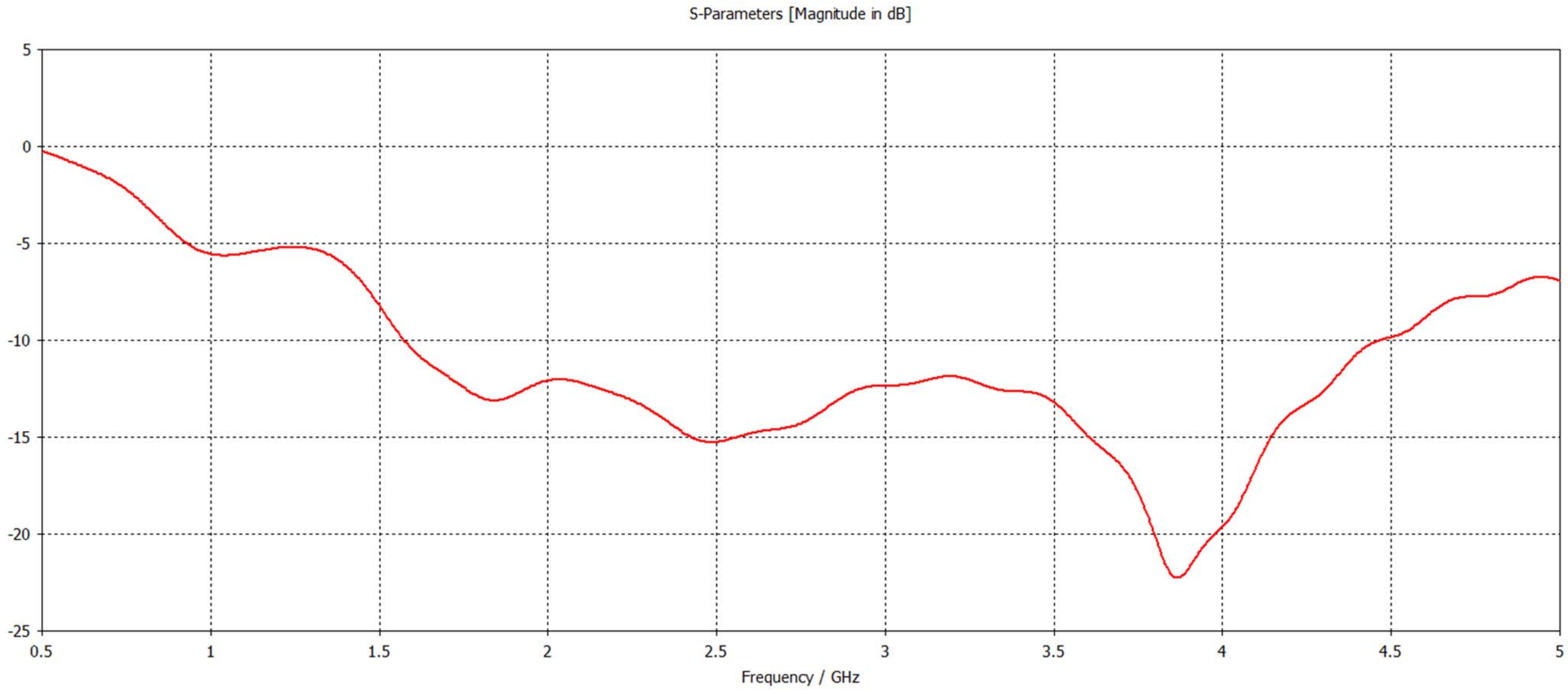
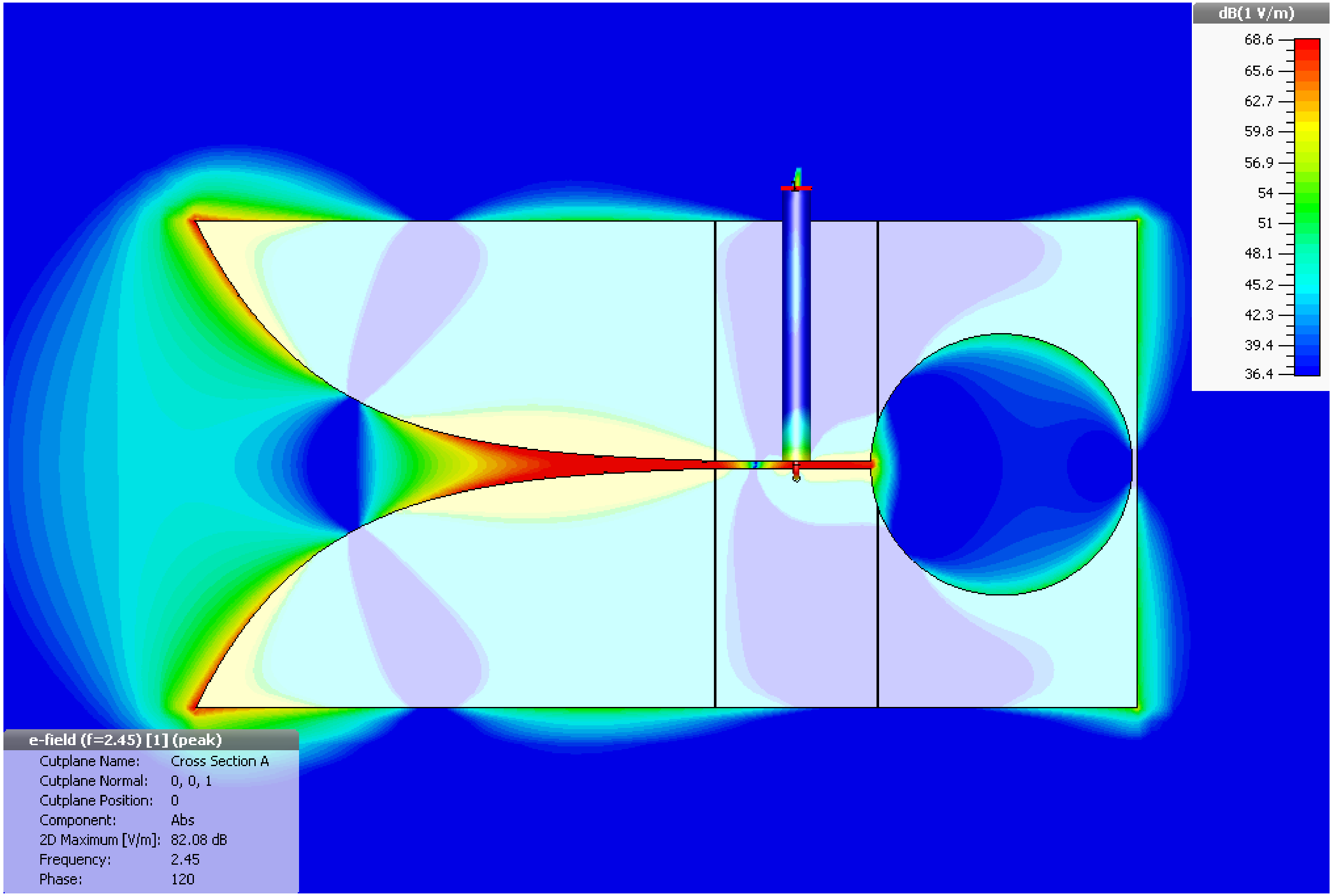
Time Domain Method – Example Matching



Finite Element Method – Example E-Field



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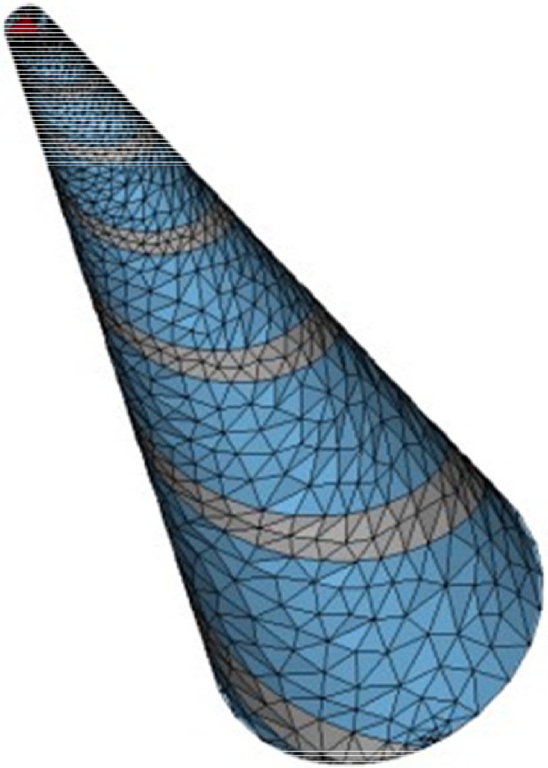
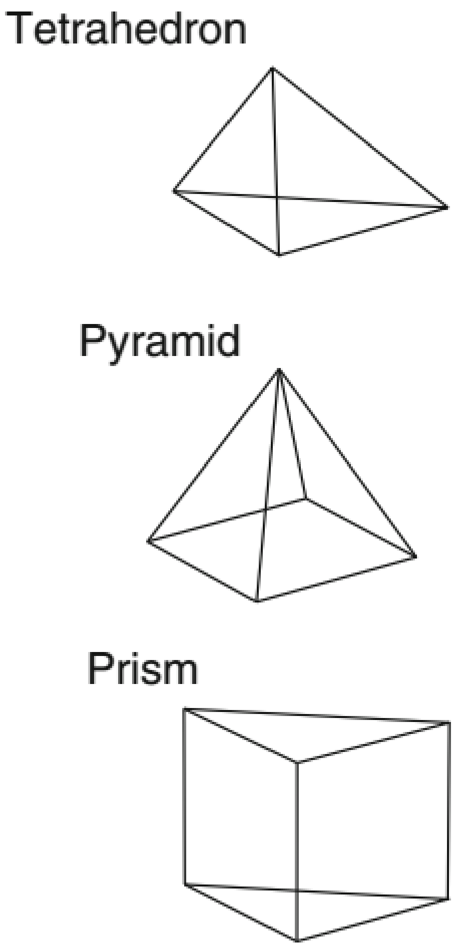
Frequency Domain Methods

- FEM - Finite Element Method
  - Full-wave
  - Simulation at single frequency points
  - Used for electrically large or inhomogeneous dielectric bodies
  - Thin dielectric sheets
  - Metallic surfaces with a finite surface impedance.
  - Metallic surfaces with an electrically thin surface coating
  - Ideal for waveguides and resonators



Finite Element Method

- Unstructured meshes (tetrahedron etc.)
  - Good representation of curved objects
  - Higher local resolution to resolve fine structures
  - Accurate meshing of arbitrarily shaped structures
- Adaptive meshing: first coarse meshing is used and then mesh is refined in regions with rapid field variations
- A linear system of equations has to be solved to update the fields, thus:
  - More computer resources needed for the same number of cells compared to FDTD
  - Poor scaling ability



Finite Element Method - Algorithm

Karlsruhe Institute of Technology

1. Subdivide the solution domain Ω into cells.
2. Approximate the solution by an expansion in a finite number of *basis functions i.e.* $f(r)$. The basis functions are generally low-order polynomials that are nonzero only in a few adjacent elements.
3. Form the residual $r = L[f] - s$, which we want to make as small as possible. In general, it will not be zero pointwise, but we require it to be zero in the so-called weak sense by setting a weighted average of it to zero.
4. Choose *weighting*, functions w_i i= 1, 2,….,n (as many as there are unknown coefficients) for weighting the residual *r*.
5. Set the weighted residuals to zero and solve for the unknowns f_i i.e., solve the set of equations $\langle w_i, , r\rangle = \int_\Omega w_i r d\Omega = 0$, i= 1, 2,….,n.

Finite Element Method – Maxwell's Equations

- Maxwell's equations in frequency domain

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t}\vec{B} \quad \Longrightarrow \quad \vec{H} = \frac{j}{\omega}\mu^{-1}\nabla \times \vec{E}$$

$$\nabla \times \vec{H} = -\frac{\partial}{\partial t}\vec{D} + \overrightarrow{J_M} + \vec{J_s} \quad \Longrightarrow \quad \nabla \times \vec{H} - j\omega\varepsilon\vec{E} + \kappa\vec{E} - \vec{J_s} = 0$$

$$\frac{j}{\omega}\nabla \times \mu^{-1}\nabla \times \vec{E} - j\omega\varepsilon\vec{E} - \kappa\vec{E} - \vec{J_s} = 0 \ /\cdot\ (-j\omega)$$

$$(\nabla \times \mu^{-1}\nabla \times -\omega^2\varepsilon + j\omega\kappa)\vec{E} = -j\omega\vec{J_s}$$

$$(\tilde{C}{M_\mu}^{-1}C - \omega^2 M_\varepsilon + j\omega M_\kappa)\hat{e} = -j\omega\hat{\vec{j_s}}$$

- Where $\overrightarrow{J_M}$ and $\vec{J_s}$ are currents in material and currents from the source

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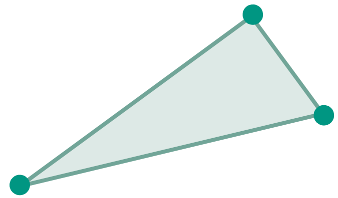
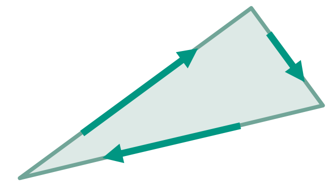
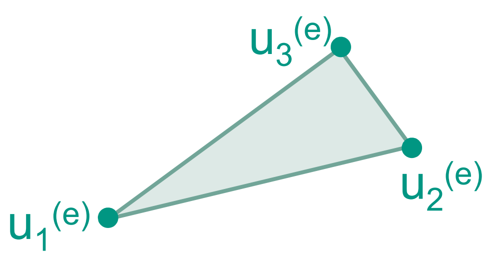
Finite Element Method – Matrix Filling

- We can either define node elements or edge elements





- We decide to take node elements

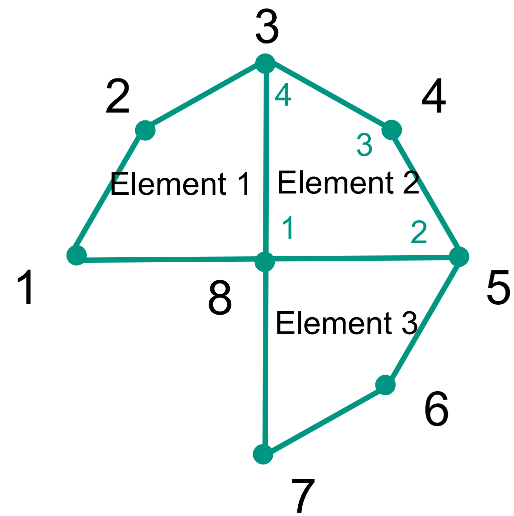


$$(\nabla \times \mu^{-1}\nabla \times -\omega^2\varepsilon + j\omega\kappa)\vec{E} = -j\omega\vec{J_s}$$

$$\underbrace{\begin{bmatrix} K_{11}^{(e)} & \cdots & K_{13}^{(e)} \\ \vdots & \ddots & \vdots \\ K_{31}^{(e)} & \cdots & K_{33}^{(e)} \end{bmatrix}}_{\text{element matrix } K^{(e)}} \begin{bmatrix} u_1^{(e)} \\ u_2^{(e)} \\ u_3^{(e)} \end{bmatrix} = \begin{bmatrix} b_1^{(e)} \\ b_2^{(e)} \\ b_3^{(e)} \end{bmatrix}$$

Finite Element Method – Matrix Filling

- Now we have to fill global matrix
  - Match local to the global node numbers
  - Add [K$^{(2)}$] to the global matrix
  - Carry on with the procedure for further elements
- The same procedure is used to fill [b] matrix



$$K = \begin{bmatrix} K_{11}^{(1)} & K_{14}^{(1)} & K_{13}^{(1)} & 0 & 0 & 0 & 0 & K_{12}^{(1)} \\ K_{41}^{(1)} & K_{44}^{(1)} & K_{43}^{(1)} & 0 & 0 & 0 & 0 & K_{42}^{(1)} \\ K_{31}^{(1)} & K_{34}^{(1)} & K_{33}^{(1)} + K_{44}^{(2)} & K_{43}^{(2)} & K_{42}^{(2)} & 0 & 0 & K_{32}^{(1)} + K_{41}^{(2)} \\ 0 & 0 & K_{34}^{(2)} & K_{33}^{(2)} & K_{32}^{(2)} & 0 & 0 & K_{31}^{(2)} \\ 0 & 0 & K_{24}^{(2)} & K_{23}^{(2)} & K_{22}^{(2)} & 0 & 0 & K_{21}^{(1)} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ K_{21}^{(1)} & K_{24}^{(1)} & K_{23}^{(1)} + K_{14}^{(2)} & K_{13}^{(2)} & K_{12}^{(2)} & 0 & 0 & K_{22}^{(1)} + K_{11}^{(2)} \end{bmatrix}$$

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Finite Element Method – Overall solution

- Now that we filled the matrixes we can solve the equation system, and get the E-field at all nodes

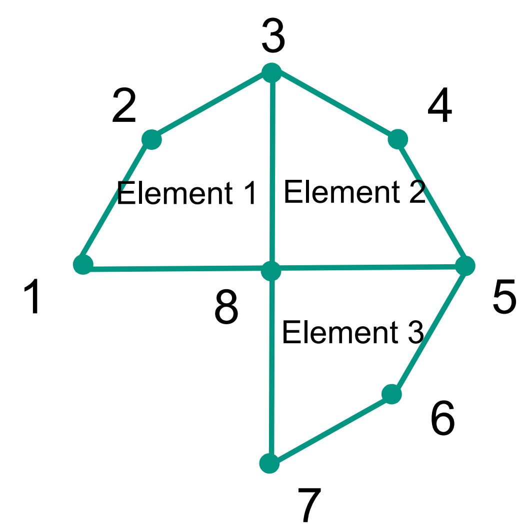
$$(\nabla \times \mu^{-1} \nabla \times -\omega^2 \varepsilon + j\omega\kappa)\vec{E} = -j\omega\vec{J_s}$$

$$\Downarrow$$

$$[K][\vec{E}] = [b]$$

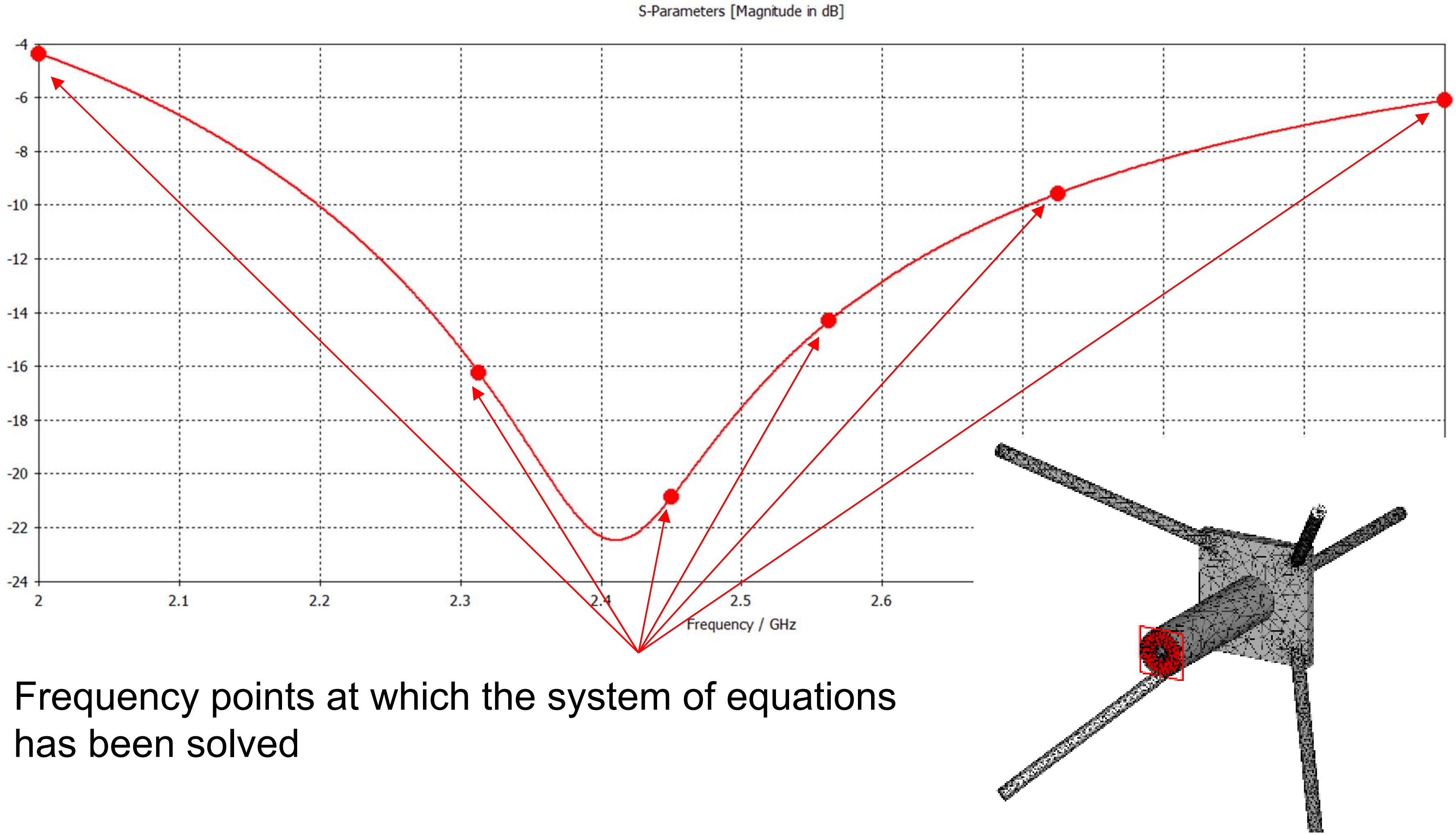
$$\Downarrow$$

$$[\vec{E}] = -j\omega[K]^{-1}[\vec{J_s}]$$



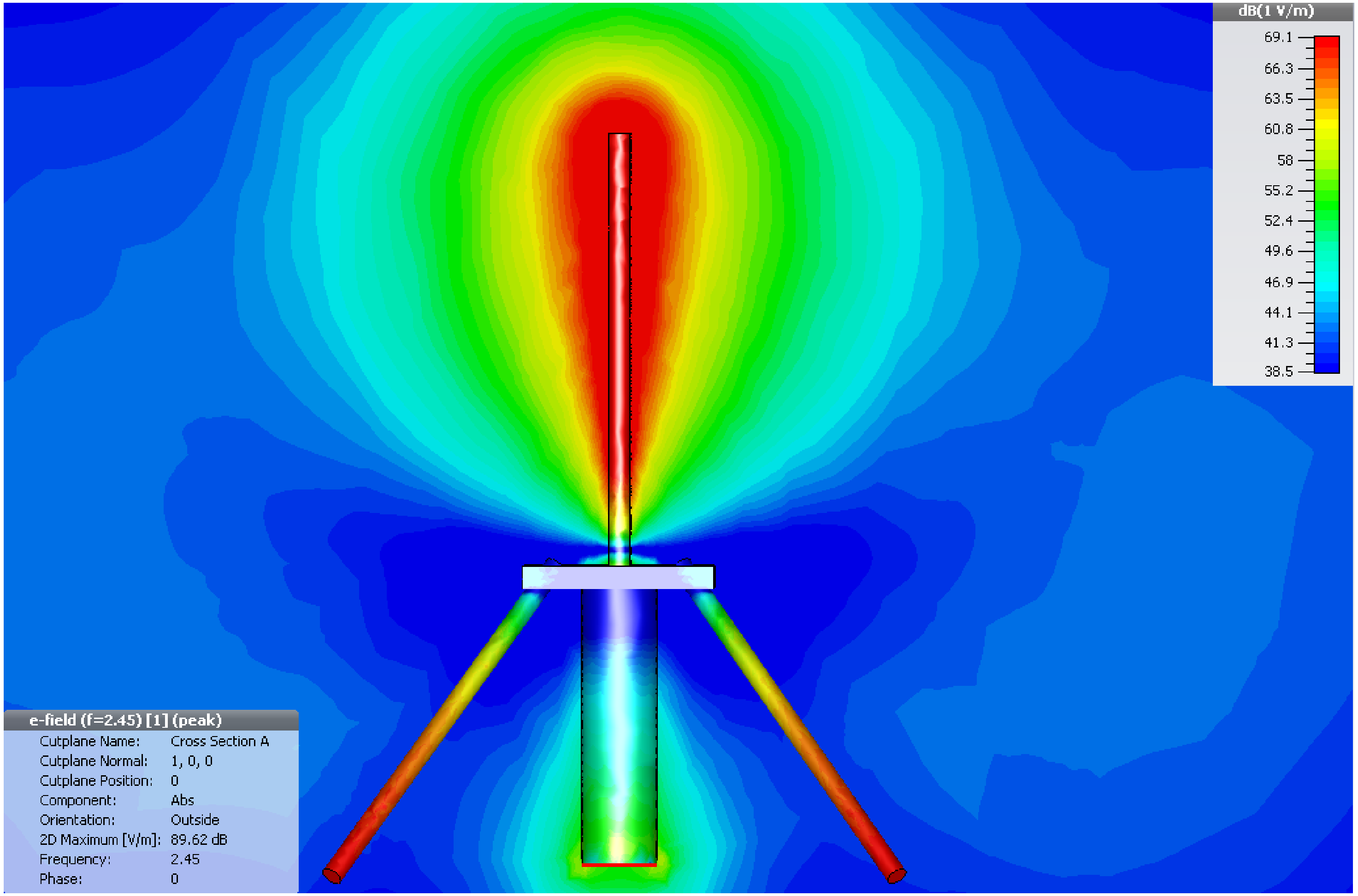
- E-field between the nodes can be approximated

Finite Element Method – Example Matching



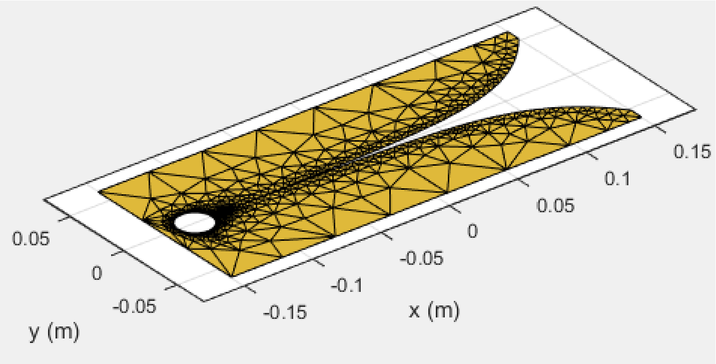
Frequency points at which the system of equations has been solved

Finite Element Method – Example E-Field



Method of Moments (Boundary Element Method)

- Problems involving currents on metallic and dielectric structures – PCB, patch antennas
- Solving scattering problems in frequency domain: RCS determination
- Similar mesh structure as in case of FEM (triangle)
- Simulation in frequency domain
- Integral form of Maxwell's equations
- MoM is a 2.5D and not 3D method (not exactly full-wave)
    - Special extensions needed for multilayer dielectric



Example of MoM implemented in Antenna ToolBox in Matlab

Method of Moments

- MoM is a procedure for solving linear equations of the form:

$$L(f) = g$$

L is a linear operator, *f* is the searched response, *g* is known excitation

- In case of antenna analysis f is the current distribution and g is the field imposed at antenna port.
- Unknown quantity *f* is expanded in in set of independent subdomain basis functions f_n, and α_j the unknown coefficients are defined

$$\vec{f} = \sum_{n=1}^{N} \alpha_n \overrightarrow{f_n}$$

- Once the currents are calculated for the whole structure, the electric/magnetic fields can be calculated for any point in space

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Method of Moments – Current Determination

- What we are looking for is an equation in form:

$$[\vec{I}] = [\vec{Z}]^{-1}[\vec{E}]$$

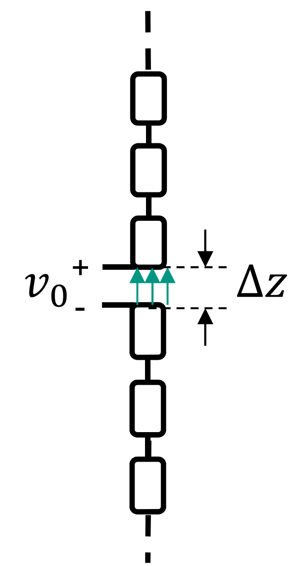
$$\overrightarrow{E_z^{inc}}(\vec{z}) = \begin{cases} \frac{v_0}{\Delta z}, at\ gap \\ 0, elswhere \end{cases}$$

- We define Pocklington's integral equation in form of

L(f)=g

$$-j\omega\varepsilon\overrightarrow{E_z^{inc}}(\vec{z}) = \boxed{\int_{-L/2}^{L/2} I_z(z')\left[\beta^2 + \frac{\partial^2}{\partial z^2}\right]\frac{e^{-j\beta r}}{4\pi r}dz'}$$

$\boxed{g=\overrightarrow{E_z^{inc}}(\vec{z})}$ $\boxed{L(f)}$ $\boxed{f = I(z)}$



Where r is defined as: $r = \sqrt{(z - z') + a^2}$

- Now we need to write the current function as set of basis functions

$$I_z(z) = \sum_{n=1}^{N} \alpha_n v_n(z)$$

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Method of Moments – Current Determination

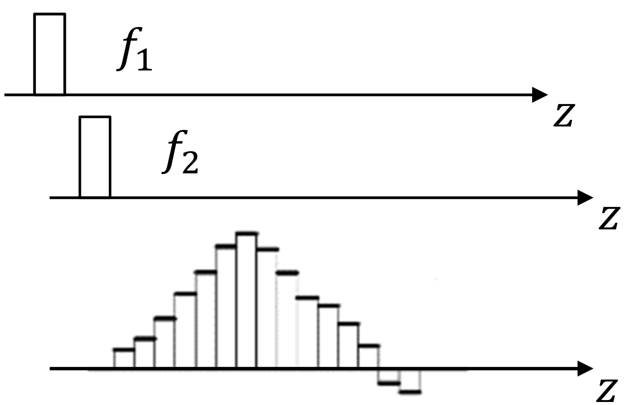
- We substitute the new current function into Pocklington's equation:

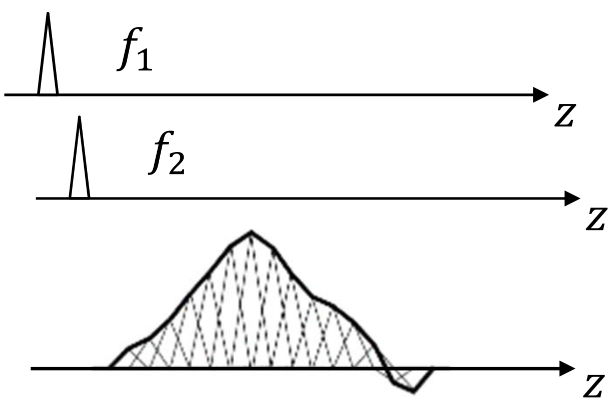
$$-j\omega\varepsilon \overrightarrow{E_z^{inc}}(\vec{z}) = \int_{-L/2}^{L/2} \left[\sum_{n=1}^{N} \alpha_n v_n(z')\right] \left[\beta^2 + \frac{\partial^2}{\partial z^2}\right] \frac{e^{-j\beta r}}{4\pi r}\, dz'$$

- We test both sides of the equation using the inner product of the basis function:

$$\left\langle v_m(z), -j\omega\varepsilon \overrightarrow{E_z^{inc}}(\vec{z}) \right\rangle = \left\langle v_m(z), \sum_n \alpha_n \int_{v_n} v_n(z') \left[\beta^2 + \frac{\partial^2}{\partial z^2}\right] \frac{e^{-j\beta r}}{4\pi r}\, dz' \right\rangle$$

$$-j\omega\varepsilon \int_{v_m} v_m(z) \overrightarrow{E_z^{inc}}(\vec{z}) = \sum_n \alpha_n \int_{v_m} v_m(z) \int_{v_n} v_n(z') \left[\beta^2 + \frac{\partial^2}{\partial z^2}\right] \frac{e^{-j\beta r}}{4\pi r}\, dz'\, dz$$





Method of Moments – Current Determination

- Having constructed a matrix equation, we can see the relation

$$-j\omega\varepsilon\int_{v_m} v_m(z)\underbrace{\overrightarrow{E_z^{inc}}(\vec{z})}_{[g_m]} = \sum_n \underbrace{\alpha_n}_{I(z)}\boxed{\int_{v_m} v_m(z)\int_{v_n} v_n(z')\left[\beta^2+\frac{\partial^2}{\partial z^2}\right]\frac{e^{-j\beta r}}{4\pi r}dz'dz}_{[z_{mn}]}$$

$$[z_{mn}][\alpha_n] = [g_m]$$

- Having solved all the problems, we can transform the matrix equation to the final form

$$[z_{mn}][i_n] = \left[-j\omega\varepsilon\overrightarrow{E_z^{inc}}(z_m)\right] \qquad \overrightarrow{E_z^{inc}}(z_m) = \frac{V_m}{\Delta z}$$

true Z $\longrightarrow$ $\boxed{\frac{j\Delta z\eta}{\beta}[z_{mn}]}[i_n] = [V_m]$

$$[i_n] = [Y][V_m]$$

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Method of Moments - Algorithm

Build impedance matrix.

Transform to true impedance. $$[Z] = \frac{j\Delta z\eta}{\beta}[z_{mn}]$$

Compute admittance matrix. $$[Y] = [Z]^{-1}$$

Compute source voltage. $$[v_m]^T = [0 \cdots \quad 1 \quad \cdots 0]$$

Calculate current. $$[i] = [z_{mn}]^{-1}[v]$$

Compute input impedance (fed at segment n). $$Z_{in} = {}^{v_n}/_{i_n}$$

Compute pattern and gain

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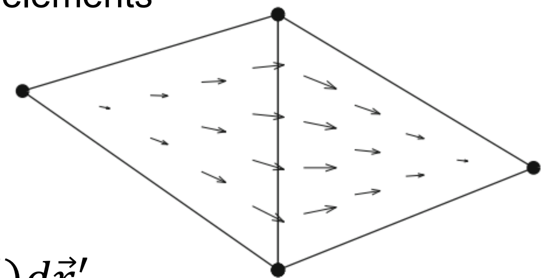
Method of Moments – Field Calculation

- Surface structures are divided into more complex basis functions: triangular patches or quadrilaterals
- A system of equations is established that involves the sources and Green‘s functions
- Electric Field Integral Equation (EFIE)

Rao-Wilton-Glisson basis function extending over two triangular elements



$$\vec{E}(\vec{r}) = -j\omega\mu \int_S \boxed{G(\vec{r},\vec{r}')} J(\vec{r}) d\vec{r}' - \frac{1}{j\omega\varepsilon} \nabla \int_S \boxed{G(\vec{r},\vec{r}')} \nabla' \cdot J(\vec{r}') d\vec{r}'$$

Green‘s function

$$G(\vec{r},\vec{r}') = \frac{1}{4\pi|r - r'|}$$

- However the EFIE is hard to solve numerically, due to the singularity of Green’s function
- Solution: combine EFIE and MFIE in CFIE (combined field integral equation)

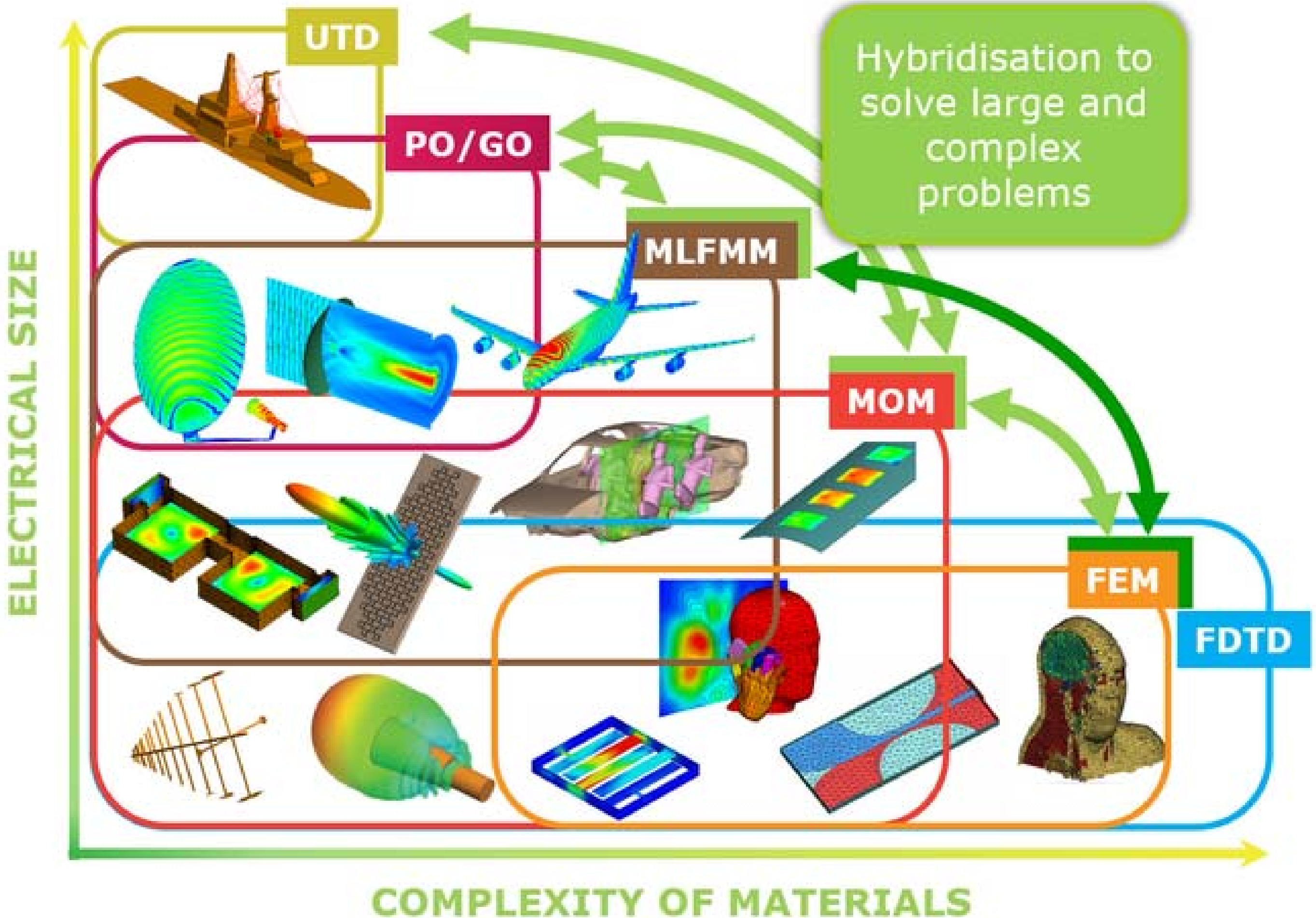
Numerical Methods – Overview

- Problem scaling:
  - FDTD: O(n)~N
  - FEM: O(n)~ N^2
  - MoM: O(n)~ N^3

Other techniques needed for large simulation problems

- UTD – Uniform Theory of Diffraction
  - Approximation of EM fields as quasi optical for determination diffraction coefficients
  - Electrically extremely large structures
- GO/PO – Geometrical Optics / Physical Optics
  - GO is ray based and considers all propagation phenomena (reflection, refraction etc.)
  - PO is very similar to UTD however bases on currents and not on rays
  - Electrically very large structures
- MLFMM – Multilevel Fast Multipole Method
  - Full-wave method based on multipole expansion
  - Requires less memory and processing power than MoM

Numerical Methods – Overview



Quelle: www.feko.info

Electromagnetic Modeling Software

- Simulation software supports design process
  - Lower number of iterations
  - Lower time and cost
  - Better insight into structure behavior

- Software solutions available on the market
  - CST
  - HFSS
  - FEKO
  - EMPIRE
  - EMPro
  - ADS MOMENTUM

Depending on the prototype’s structure and size different numerical methods are applicable

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